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DISTRIBUTION OF PRIMES AND OF INTERVAL PRIME PAIRS BASED ON Θ FUNCTION

YIFANG FAN, ZHIYU LI

ABSTRACT. Θ function is defined based upon Kronecher symbol. In light of the principle of inclusion-exclusion, Θ function of sine function is used to denote the distribution of composites and primes. The structure of Goldbach Conjecture has been analyzed, and Ξ function is brought forward by the linear diophantine equation; by relating to Θ function, the interval distribution of composite pairs and prime pairs (i.e. the Goldbach Conjecture) is thus obtained. In the end, Abel's Theorem (Multiplication of Series) is used to discuss the lower limit of the distribution of the interval prime pairs.

1. INTRODUCTION

Distribution of primes has long been one of the main issues of number theory research. Gauss and Legendre presented the issue of the distribution of primes in the form of conjecture; in 1837, Dirichlet proved the distribution of primes in Arithmetic Progressions [1]; in 1896, both Hadamard and Poussin proved the basic laws of the distribution of primes - the prime number theorem; in 1948, Selberg and Erdös both proved the prime number theorem by elementary proofs [2, 3]. But still, the present prime number theorem cannot denote the exact number of primes of natural number.

In 1742, Goldbach brought forward his famous Goldbach Conjecture. Up to now, the best developments are: in 1937, Vinogradov proved that each large odd number was the sum of three odd primes [2, 4]; in 1966, Chen Jingrun proved that each large even number can be denoted as the sum of one prime number and the product of no more than two primes to multiply [5, 6]. In this paper, issues of the distribution of primes and of the interval prime pairs are discussed by defining Θ function and Ξ function. Hereof, p,q,r stand for primes, and $\lceil x \rceil$ is the greatest integer less than or equal to x. A couple of important theorems referred in this paper include:

Theorem 1. If N is a composite and p is its smallest positive divisor, then [7]

$$p \leq \sqrt{N}$$

Theorem 2. (Prime Number Theorem) Let N be composite and $\pi(N)$ denote the number of primes smaller than, it follows that [2, 7]

$$\lim \pi(N) = \frac{N}{\ln N}$$

Key words and phrases. Θ function, Distribution of primes, Distribution of composites, Ξ function, Distribution of composite pairs, Distribution of prime pairs.

Theorem 3. (Principle of Inclusion-Exclusion). Suppose that the number of elements is n, where element n_α has the property of α, and element n_β has that of β, element $n_{\alpha\beta}$ has the properties of both $\alpha, \beta \cdots$, element $n_{\alpha\beta\gamma}$ has all the properties of $\alpha, \beta, \gamma \cdots$, then the number of elements possessing none of the $\alpha, \beta, \gamma \cdots$ properties is[2, p9; 7, p525]

$$n - n_\alpha - n_\beta - n_\gamma - \cdots + n_{\alpha\beta} + \cdots - n_{\alpha\beta\gamma} - \cdots + \cdots - \cdots$$

2. Distribution of primes

Prime number theorem has presented the basic properties of the distribution of primes. As for a natural number, the result from the prime number theorem may not be an exact value, but the error will be diminished as N becomes a larger number [2]. Some definitions and theorems will be employed to illustrate this function.

2.1. **Use $\lceil x \rceil$ to denote the distribution of primes and composites. Theorem 4.** When N is an even composite, and $\pi(N)$ denotes the number of primes no greater than N, then it shows that [7, 8]
(2.1.1)

$$\pi(N) = N - 1 - \sum_{i=1}^{l} \left\lceil \frac{N-p_i}{p_i} \right\rceil + \sum_{i=1}^{l-1} \sum_{j=i+1}^{l} \left\lceil \frac{N}{p_i p_j} \right\rceil - \sum_{i=1}^{l-2} \sum_{j=i+1}^{l-1} \sum_{k=j+1}^{l} \left\lceil \frac{N}{p_i p_j p_k} \right\rceil + \cdots - \cdots$$

Where $p_i \leq \sqrt{N}$, $l = \pi(\sqrt{N})$and $p_i p_j \leq N, p_i p_j p_k \leq N \cdots$

Proof. By Theorem 3, it shows that
(2.1.2)

$$\pi(N) = N - 1 + \pi(\sqrt{N}) - \sum_{i=1}^{l} \left\lceil \frac{N}{p_i} \right\rceil + \sum_{i=1}^{l-1} \sum_{j=i+1}^{l} \left\lceil \frac{N}{p_i p_j} \right\rceil - \sum_{i=1}^{l-2} \sum_{j=i+1}^{l-1} \sum_{k=j+1}^{l} \left\lceil \frac{N}{p_i p_j p_k} \right\rceil + \cdots - \cdots$$

By Theorem 1, for $p_i \leq \sqrt{N}$, it shows that

$$\sum_{i=1}^{l} \left\lceil \frac{N-p_i}{p_i} \right\rceil = \sum_{i=1}^{l} \left\lceil \frac{N}{p_i} \right\rceil - \pi(\sqrt{N}) \tag{2.1.3}$$

Thus the theorem is proved.

Theorem 5. When N is an even composite, and $\varpi(N)$ denotes the number of composites no greater than N, it shows that
(2.1.4)

$$\varpi(N) = \sum_{i=1}^{l} \left\lceil \frac{N-p_i}{p_i} \right\rceil - \sum_{i=1}^{l-1} \sum_{j=i+1}^{l} \left\lceil \frac{N}{p_i p_j} \right\rceil + \sum_{i=1}^{l-2} \sum_{j=i+1}^{l-1} \sum_{k=j+1}^{l} \left\lceil \frac{N}{p_i p_j p_k} \right\rceil - \cdots + \cdots$$

Proof. By the definition of positive integer, it follows that

$$N = \varpi(N) + \pi(N) + 1 \tag{2.1.5}$$

By Theorem 4, the proposition is proved.

2.2. Θ **function. Definition 1.** Θ function has the following properties as long as real numbers are concerned

$$\Theta(x) = \begin{cases} 1 & x = 0 \\ 0 & x \neq 0 \end{cases} \tag{2.2.1}$$

Let

$$\Theta(x) = \delta_{ix}$$

Where δ_{ix} is Kronecher symbol, that is

$$\delta_{ix} = \begin{cases} 1 & i = x \\ 0 & i \neq x \end{cases}$$

Suppose $i = 0$, then $\Theta(x)$ can be denoted as

$$\Theta(x) = \delta_{0x}$$

Θ function shows that when $x = 0$, the Θ functional value of Θ is 1; in other cases, it is 0. Θ is a given Kronecher symbol.

2.3. **Use Θ function of sine function to denote the distribution of primes and that of composites.** For an even composite, $\lceil x \rceil$ of Theorem 4 and Theorem 5 can be substituted by Θ function of Definition 1.

Theorem 6. When N is an even composite, use $\varpi_p(N)$ to denote the number of composites in N that can be exactly divided by prime p, then

$$\varpi_p(N) = \left\lceil \frac{N-p}{p} \right\rceil = \left\lceil \frac{N}{p} \right\rceil - 1 \tag{2.3.1}$$

Θ function of sine function can be denoted as

$$\varpi_p(N) = \sum_{x=1}^{N} \Theta\left(\sin\left(\frac{x\pi}{p}\right)\right) - 1 \tag{2.3.2}$$

Proof. By the properties of sine function, it follows that

$$\sin\left(\frac{x\pi}{p}\right)_{1 \leqslant x \leqslant p} = (y_1, y_2, \cdots, y_p) \tag{2.3.3}$$

When x is positive integer, then

$$\begin{cases} y_1 \neq 0, y_2 \neq 0, \cdots, y_{p-1} \neq 0 \\ y_p = 0 \end{cases} \tag{2.3.4}$$

By Definition 1

$$\Theta\left(\sin\left(\frac{x\pi}{p}\right)_{1 \leqslant x \leqslant p}\right) = \Theta\left(y_1, y_2, \cdots, y_p\right) = (0, 0, \cdots, 1) \tag{2.3.5}$$

That is

$$\sum_{x=1}^{p} \Theta\left(\sin\left(\frac{x\pi}{p}\right)\right) = 1 \tag{2.3.6}$$

In the same way, it shows that

$$\sum_{x=p+1}^{2p} \Theta\left(\sin\left(\frac{x\pi}{p}\right)\right) = 1, \cdots, \sum_{x=(n-1)p+1}^{np} \Theta\left(\sin\left(\frac{x\pi}{p}\right)\right) = 1 \tag{2.3.7}$$

Where n is a positive integer and $np \leqslant N$.
Therefore

$$\sum_{x=1}^{N} \Theta\left(\sin\left(\frac{x\pi}{p}\right)\right) = \left\lceil \frac{N}{p} \right\rceil$$

and by eq.(7), it follows that

Theorem 7. When N is an even composite, use $\varpi_{pq}(N)$ to denote the number of composites in N that can be exactly divided by primes p, q, then

$$\varpi_{pq}(N) = \left\lceil \frac{N}{p} \right\rceil + \left\lceil \frac{N}{p} \right\rceil - \left\lceil \frac{N}{pq} \right\rceil - 2 \tag{2.3.8}$$

Θ function of sine function can be denoted as
(2.3.9)

$$\varpi_{pq}(N) = \sum_{x=1}^{N} \Theta\left(\sin\left(\frac{x\pi}{p}\right)\right) + \sum_{x=1}^{N} \Theta\left(\sin\left(\frac{x\pi}{q}\right)\right) - \sum_{x=1}^{N} \Theta\left(\sin\left(\frac{x\pi}{pq}\right)\right) - 2$$

Proof. By the properties of sine function, it follows that

$$\sin\left(\frac{x\pi}{pq}\right)_{1\leqslant x\leqslant pq} = (y_1, y_2, \cdots, y_{pq}) \tag{2.3.10}$$

When x is positive integer, it follows that

$$\begin{cases} y_1 \neq 0, y_2 \neq 0, \cdots, y_{pq-1} \neq 0 \\ y_{pq} = 0 \end{cases} \tag{2.3.11}$$

By Definition 1

$$\Theta\left(\sin\left(\frac{x\pi}{pq}\right)_{1\leqslant x\leqslant pq}\right) = \Theta\left(y_1, y_2, \cdots, y_{pq}\right) = (0, 0, \cdots, 1) \tag{2.3.12}$$

That is

$$\sum_{x=1}^{pq} \Theta\left(\sin\left(\frac{x\pi}{pq}\right)\right) = 1 \tag{2.3.13}$$

In the same way, it shows that

$$\sum_{x=p+1}^{2pq} \Theta\left(\sin\left(\frac{x\pi}{pq}\right)\right) = 1, \cdots, \sum_{x=(n-1)pq+1}^{npq} \Theta\left(\sin\left(\frac{x\pi}{pq}\right)\right) = 1 \tag{2.3.14}$$

Where n is a positive integer and $npq \leqslant N$.
Therefore

$$\sum_{x=1}^{N} \Theta\left(\sin\left(\frac{x\pi}{pq}\right)\right) = \left\lceil \frac{N}{pq} \right\rceil \tag{2.3.15}$$

By Theorem 6, the theorem is proved.

Theorem 8. When N is an even composite, use $\varpi_{pq}(N)$ to denote the number of composites in N that can be exactly divided by primes p, q, then Θ function of sine function can be denoted as

$$\varpi_{pq}(N) = \sum_{x=1}^{N} \Theta\left(\sin\left(\frac{x\pi}{p}\right)\sin\left(\frac{x\pi}{q}\right)\right) - 2 \tag{2.3.16}$$

Proof. Since

$$\left(\sin\left(\frac{x\pi}{p}\right)\right) = 0 \Longrightarrow \Theta\left(\sin\left(\frac{x\pi}{p}\right)\sin\left(\frac{x\pi}{q}\right)\right) = 1 \tag{2.3.17}$$

$$\left(\sin\left(\frac{x\pi}{q}\right)\right) = 0 \Longrightarrow \Theta\left(\sin\left(\frac{x\pi}{p}\right)\sin\left(\frac{x\pi}{q}\right)\right) = 1 \tag{2.3.18}$$

(2.3.19)
$$\left(\sin\left(\frac{x\pi}{p}\right)\right)=0, and \left(\sin\left(\frac{x\pi}{q}\right)\right)=0 \Longrightarrow \Theta\left(\sin\left(\frac{x\pi}{p}\right)\sin\left(\frac{x\pi}{q}\right)\right)=1$$

Therefore, it follows that

(2.3.20)
$$\begin{aligned}\sum_{x=1}^{N}\Theta\left(\sin\left(\frac{x\pi}{p}\right)\sin\left(\frac{x\pi}{q}\right)\right) &= \sum_{x=1}^{N}\Theta\left(\sin\left(\frac{x\pi}{p}\right)\right)+\sum_{x=1}^{N}\Theta\left(\sin\left(\frac{x\pi}{q}\right)\right)\\ &-\sum_{x=1}^{N}\Theta\left(\sin\left(\frac{x\pi}{pq}\right)\right)\\ &=\left\lceil\frac{N}{p}\right\rceil+\left\lceil\frac{N}{q}\right\rceil-\left\lceil\frac{N}{pq}\right\rceil\end{aligned}$$

By Theorem 7, the proposition is proved.

Theorem 9. When N is an even composite, $p, q, \cdots, s \leqslant \sqrt{N}$, and $p\times q\times\cdots\times s \leqslant N$, then.

$$\left\lceil\frac{N}{pq\cdots s}\right\rceil=\sum_{x=1}^{N}\Theta\left(\sin\left(\frac{x\pi}{pq\cdots s}\right)\right) \tag{2.3.21}$$

Proof. By the properties of sine function, it follows that

$$\sin\left(\frac{x\pi}{pq\cdots s}\right)_{1\leqslant x\leqslant pq\cdots s}=(y_1, y_2, \cdots, y_{pq\cdots s}) \tag{2.3.22}$$

When x is positive integer, it follows that

$$\begin{cases} y_1\neq 0, y_2\neq 0, \cdots, y_{pq\cdots s-1}\neq 0\\ y_{pq\cdots s}=0\end{cases} \tag{2.3.23}$$

By Definition 1

$$\Theta\left(\sin\left(\frac{x\pi}{pq\cdots s}\right)_{1\leqslant x\leqslant pq\cdots s}\right)=\Theta\left(y_1, y_2, \cdots, y_{pq\cdots s}\right)=(0, 0, \cdots, 1) \tag{2.3.24}$$

That is

$$\sum_{x=1}^{pq\cdots s}\Theta\left(\sin\left(\frac{x\pi}{pq\cdots s}\right)\right)=1 \tag{2.3.25}$$

In the same way, it follows that

(2.3.26)
$$\sum_{x=pq\cdots s+1}^{2pq\cdots s}\Theta\left(\sin\left(\frac{x\pi}{pq\cdots s}\right)\right)=1, \sum_{x=2pq\cdots s+1}^{3pq\cdots s}\Theta\left(\sin\left(\frac{x\pi}{pq\cdots s}\right)\right)=1, \cdots,$$
$$\sum_{x=(n-1)pq\cdots s+1}^{npq\cdots s}\Theta\left(\sin\left(\frac{x\pi}{pq\cdots s}\right)\right)=1$$

Therefore, the proposition is proved.

Theorem 10. When N is an even composite, $\varpi(N)$ of the number of composites smaller than N can be denoted as

$$\varpi(N)=\sum_{x=1}^{N}\Theta\left(\prod_{i=1}^{l}\sin\left(\frac{x\pi}{p_i}\right)\right)_{p_i\leqslant\sqrt{N},l=\pi(\sqrt{N})}-\pi(\sqrt{N}) \tag{2.3.27}$$

Proof. By Theorem 5, together with Theorem 6 and 7, can be denoted as

$$\varpi(N)=\sum_{i=1}^{l}\sum_{x=1}^{N}\Theta\left(\sin\left(\frac{x\pi}{p_i}\right)\right)_{p_i\leqslant\sqrt{N}}-\sum_{i=1}^{l-1}\sum_{j=i+1}^{l}\sum_{x=1}^{N}\Theta\left(\sin\left(\frac{x\pi}{p_ip_j}\right)\right)_{p_i\leqslant\sqrt{N},p_ip_j\leqslant N}+\cdots-\pi(\sqrt{N}) \tag{2.3.28}$$

Since

$$\prod_{i=1}^{l}\left(\sin\left(\frac{x\pi}{p_i}\right)\right)_{p_i\leqslant\sqrt{N}}=\left(\sin\left(\frac{x\pi}{p_1}\right)\sin\left(\frac{x\pi}{p_2}\right)\cdots\sin\left(\frac{x\pi}{p_l}\right)\right)_{l=\pi(\sqrt{N})} \tag{2.3.29}$$

by Theorem 8 and 9, it follows that

$$\sum_{i=1}^{N}\Theta\left(\prod_{i=1}^{l}\sin\left(\frac{x\pi}{p_i}\right)\right)_{p_i\leqslant\sqrt{N}}=\sum_{i=1}^{l}\sum_{x=1}^{N}\Theta\left(\sin\left(\frac{x\pi}{p_i}\right)\right)_{p_i\leqslant\sqrt{N}}-\sum_{i=1}^{l-1}\sum_{j=i+1}^{l}\sum_{x=1}^{N}\Theta\left(\sin\left(\frac{x\pi}{p_ip_j}\right)\right)_{p_i\leqslant\sqrt{N},p_ip_j\leqslant N}+\cdots \tag{2.3.30}$$

Therefore, the proposition is proved.

Theorem 11. When N is an even composite, use $\pi(N)$ to denote the number of primes smaller than N, then

$$\pi(N)=N-1-\sum_{x=1}^{N}\Theta\left(\prod_{i=1}^{l}\sin\left(\frac{x\pi}{p_i}\right)\right)_{p_i\leqslant\sqrt{N},l=\pi(\sqrt{N})}+\pi(\sqrt{N}) \tag{2.3.31}$$

Proof. When the distribution of composites is determined, that of the primes can thus be obtained. By Theorem 4 and Theorem 10, prime distribution function is obtained.

3. Distribution of interval prime pairs

3.1. Ξ **function.** Take the integer N as an example. We can interpret the Goldbach Conjecture as follows:

$$N=x+y \tag{3.1.1}$$

The so-called Goldbach Conjecture is that the two numbers that make up the integer N are both primes. Such primes are called "prime pair". Now let's denote the continuous positive integers of $1,2,3,\cdots,N-3,N-2,N-1$ and $N-1,N-2,N-3,\cdots,3,2,1$ respectively with functions. Let's use a function to analyze the solution of prime pairs of the linear diophantine equation.

Definition 2. $\overrightarrow{\Gamma}$ function, $\overleftarrow{\Gamma}$ function and Ξ function

Suppose that $\overrightarrow{\Gamma}(z)$ is the ordered function of continuous positive integer of $(1,N)$ and $\overleftarrow{\Gamma}(z)$ is the ordered function of continuous positive integer of $(N,1)$, then

$\overrightarrow{\Gamma}(z)+\overleftarrow{\Gamma}(z)$ will be named as Ξ function of the ordered series of $\overrightarrow{\Gamma}(z)$ and $\overleftarrow{\Gamma}(z)$,$\overrightarrow{\Gamma}(z)$ and $\overleftarrow{\Gamma}(z)$ would be an object to each other. Use $\Xi(z)$ to denote the solution set of two ordered functions, with possessing the following basic properties:

(1) Abort $\overrightarrow{\Gamma}(z)$ and $\overleftarrow{\Gamma}(z)$, we have

$$\overrightarrow{\Gamma}(z)=x, \overleftarrow{\Gamma}(z)=N-x=y \tag{3.1.2}$$

(2) The solution set of the ordered function of continuous positive integers of (1,N) is

$$\Xi(z)=\overrightarrow{\Gamma}(z)+\overleftarrow{\Gamma}(z)=N \tag{3.1.3}$$

(3) Of the results making up Ξ function, two objects will only come up with the following pairings: 1 and composite/prime, composite and composite, composite and prime, prime and composite, prime and prime, composite/prime and 1. Let's call the Ξ result making up of prime and prime as the prime pair (or Goldbach Conjecture), and the rest of them as composite pairs.

Theorem 12. $\overrightarrow{\varpi}_p(N)$ denotes the number of composites of $\overrightarrow{\Gamma}(z)$ that can be exactly divided by p and $\overleftarrow{\varpi}_p(N)$ denotes the number of composites of $\overleftarrow{\Gamma}(z)$ that can be exactly divided by p. Then for $\overrightarrow{\varpi}_p(N)$ and $\overleftarrow{\varpi}_p(N)$, it follows that

$$\overrightarrow{\varpi}_p(N)=\overleftarrow{\varpi}_p(N) \tag{3.1.4}$$

Proof. By Theorem 6, for $\overrightarrow{\varpi}_p(N)$, it follows that

$$\overrightarrow{\varpi}_p(N)=\sum_{x=1}^{N}\Theta\left(\sin\left(\frac{x\pi}{p}\right)\right)-1 \tag{3.1.5}$$

For $\overleftarrow{\varpi}_p(N)$, it follows that

$$\overleftarrow{\varpi}_p(N)=\sum_{x=N}^{1}\Theta\left(\sin\left(\frac{x\pi}{p}\right)\right)-1 \tag{3.1.6}$$

The theorem is thus proved.

Theorem 13. For the objects of $\overrightarrow{\Gamma}(N)$ and $\overleftarrow{\Gamma}(N)$, it follows that

$$\overrightarrow{\varpi}(N)=\overleftarrow{\varpi}(N) \tag{3.1.7}$$

$$\overrightarrow{\pi}(N)=\overleftarrow{\pi}(N) \tag{3.1.8}$$

Proof. By Theorem 10, it follows that

$$\overrightarrow{\varpi}(N)=\sum_{x=1}^{N}\Theta\left(\sin\left(\frac{x\pi}{p_1}\right)\sin\left(\frac{x\pi}{p_2}\right)\cdots\sin\left(\frac{x\pi}{p_l}\right)\right)_{p_i\leqslant\sqrt{N},l=\pi(\sqrt{N})}-\pi(\sqrt{N}) \tag{3.1.9}$$

$$\overleftarrow{\varpi}(N)=\sum_{x=N}^{1}\Theta\left(\sin\left(\frac{x\pi}{p_1}\right)\sin\left(\frac{x\pi}{p_2}\right)\cdots\sin\left(\frac{x\pi}{p_l}\right)\right)_{p_i\leqslant\sqrt{N},l=\pi(\sqrt{N})}-\pi(\sqrt{N}) \tag{3.1.10}$$

By Theorem 12, the theorem is proved.

3.2. **Characteristics of Ξ function.** The relationship of $\overrightarrow{\Gamma}(z)$, $\overleftarrow{\Gamma}(z)$ and $\Xi(z)$ can be illustrated by the following figure (1):

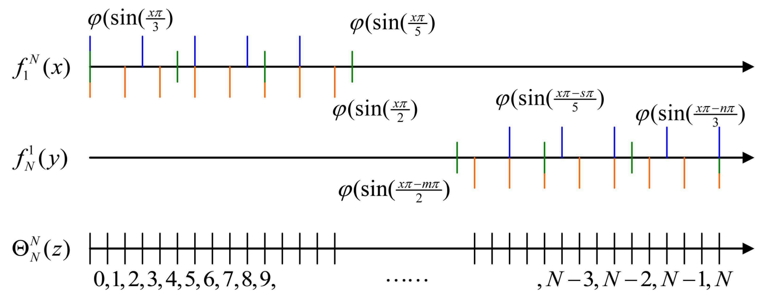

FIGURE 1.

By Figure (1), we can obtain some characteristics of Ξ function:

Theorem 14. If $N \equiv m(\mathrm{mod} p)$, for $0 < m < p$, it follows that

$$\overrightarrow{\varpi}_p(N) = \sum_{\substack{x=1 \\ p\leqslant\sqrt{N}}}^{N} \Theta\left(\sin\left(\frac{x\pi}{p}\right)\right) = \overleftarrow{\varpi}_p(N) = \sum_{\substack{x=1 \\ p\leqslant\sqrt{N}, N\equiv m(\mathrm{mod} p)}}^{N} \Theta\left(\sin\left(\frac{x\pi - m\pi}{p}\right)\right) \tag{3.2.1}$$

Proof. When $0 < m < p$, by function defined by $\Xi(x)$, for $\overrightarrow{\Gamma}(x)$ and $\overleftarrow{\Gamma}(x)$, it follows that

(3.2.2)

$$\sum_{x=1}^{N} \Theta\left(\sin\left(\frac{x\pi}{p}\right)\right) = \sum_{x=1}^{N} \Theta\left(\sin\left(\frac{x\pi - \pi}{p}\right)\right) = \cdots = \sum_{x=1}^{N} \Theta\left(\sin\left(\frac{x\pi - m\pi}{p}\right)\right)$$

In addition

(3.2.3)

$$\sum_{x=N}^{1} \Theta\left(\sin\left(\frac{x\pi}{p}\right)\right) = \sum_{x=1}^{N} \Theta\left(\sin\left(\frac{N\pi - x\pi}{p}\right)\right) = \sum_{x=1}^{N} \Theta\left(\sin\left(\frac{x\pi - m\pi}{p}\right)\right)$$

Therefore

$$\sum_{\substack{x=1 \\ p\leqslant\sqrt{N}}}^{N} \Theta\left(\sin\left(\frac{x\pi}{p}\right)\right) = \sum_{\substack{x=1 \\ p\leqslant\sqrt{N}, N\equiv m(\mathrm{mod} p)}}^{N} \Theta\left(\sin\left(\frac{x\pi - m\pi}{p}\right)\right) \tag{3.2.4}$$

Together with Theorem 13, the theorem is thus proved.

Theorem 15. When n, m are integers, it follows that

$$\Theta\left(m \sin^n\left(\frac{x\pi}{p}\right)\right) = \Theta\left(\sin\left(\frac{x\pi}{p}\right)\right) \tag{3.2.5}$$

Proof. When

$$\left(\sin\left(\frac{x\pi}{p}\right)\right) \neq 0$$

it follows that

$$\left(m \sin^n\left(\frac{x\pi}{p}\right)\right) \neq 0$$

When

$$\left(\sin\left(\frac{x\pi}{p}\right)\right) = 0$$

it shows that

$$\left(m\sin^n\left(\frac{x\pi}{p}\right)\right) = 0$$

Hence the proposition holds true.

3.3. **Distribution of composite pairs of Ξ function.** **Theorem 16.** Let

$$x = \Theta\left(\sin\left(\frac{z\pi}{p}\right)\right), y = \Theta\left(\sin\left(\frac{z\pi}{q}\right)\right)$$

for Θ function, it follows that

$$\Theta(xy) = \Theta(x) + \Theta(y) - \Theta(x+y) \tag{3.3.1}$$

Proof. When p, q are primes, the right side shows that

$$\Theta(xy) = \begin{cases} 1 & xy = 0 \\ 0 & xy \neq 0 \end{cases} \tag{3.3.2}$$

While the left side shows that

$$\begin{cases} xy \neq 0 \Rightarrow x \neq 0, y \neq 0 \Rightarrow \Theta(x) = 0, \Theta(y) = 0, \Theta(x+y) = 0 \\ xy = 0 \begin{cases} x = 0, y \neq 0 \Rightarrow \Theta(x) = 1, \Theta(y) = 0, \Theta(x+y) = 0 \\ x \neq 0, y = 0 \Rightarrow \Theta(x) = 0, \Theta(y) = 1, \Theta(x+y) = 0 \\ x = 0, y = 0 \Rightarrow \Theta(x) = 1, \Theta(y) = 1, \Theta(x+y) = 1 \end{cases} \end{cases} \tag{3.3.3}$$

Both sides are equal. Hence the theorem is proved.
Theorem 17. For Θ function, it follows that

$$\Theta\left(\sin\left(\frac{x\pi}{p}\right)\sin\left(\frac{x\pi - m\pi}{p}\right)\right)_{p\leqslant\sqrt{N};N\equiv m(\mathrm{mod}p)} = \begin{cases} \Theta\left(\sin\left(\frac{x\pi}{p}\right)\right)_{p\leqslant\sqrt{N};p|N} \\ 2\Theta\left(\sin\left(\frac{x\pi}{p}\right)\right)_{p\leqslant\sqrt{N};p\nmid N} \end{cases} \tag{3.3.4}$$

Proof. When $m = 0$ that is $p|N$, then

$$\Theta\left(\sin\left(\frac{x\pi}{p}\right)\sin\left(\frac{x\pi - m\pi}{p}\right)\right)_{p\leqslant\sqrt{N};p|N} = \Theta\left(\sin\left(\frac{x\pi}{p}\right)\sin\left(\frac{x\pi}{p}\right)\right)_{p\leqslant\sqrt{N};p|N} \tag{3.3.5}$$

By theorem 15, it shows that

$$\Theta\left(\sin\left(\frac{x\pi}{p}\right)\sin\left(\frac{x\pi}{p}\right)\right)_{p\leqslant\sqrt{N}} = \Theta\left(\sin\left(\frac{x\pi}{p}\right)\right)_{p\leqslant\sqrt{N}} \tag{3.3.6}$$

Or by Theorem 16, it follows that

$$\begin{aligned} \Theta\left(\sin\left(\frac{x\pi}{p}\right)\sin\left(\frac{x\pi - m\pi}{p}\right)\right)_{p\leqslant\sqrt{N};m=0} &= \Theta\left(\sin\left(\frac{x\pi}{p}\right)\right)_{p\leqslant\sqrt{N}} + \Theta\left(\sin\left(\frac{x\pi}{p}\right)\right)_{p\leqslant\sqrt{N}} \\ &- \Theta\left(\sin\left(\frac{x\pi}{p}\right) + \sin\left(\frac{x\pi}{p}\right)\right)_{p\leqslant\sqrt{N}} \end{aligned} \tag{3.3.7}$$

By Theorem 15, it shows that

$$\Theta\left(\sin\left(\frac{x\pi}{p}\right)+\sin\left(\frac{x\pi}{p}\right)\right)_{p\leqslant\sqrt{N}}=\Theta\left(2\sin\left(\frac{x\pi}{p}\right)\right)_{p\leqslant\sqrt{N}}=\Theta\left(\sin\left(\frac{x\pi}{p}\right)\right)_{p\leqslant\sqrt{N}} \tag{3.3.8}$$

When $m\neq 0$, that is $p\nmid N$, then by theorem 16

$$\begin{aligned}\Theta\left(\sin\left(\frac{x\pi}{p}\right)\sin\left(\frac{x\pi-m\pi}{p}\right)\right)_{p\leqslant\sqrt{N};p\nmid N;m\neq 0}&=\Theta\left(\sin\left(\frac{x\pi}{p}\right)\right)_{p\leqslant\sqrt{N}}+\Theta\left(\sin\left(\frac{x\pi-m\pi}{p}\right)\right)_{p\leqslant\sqrt{N}}\\&\quad-\Theta\left(\sin\left(\frac{x\pi}{p}\right)+\sin\left(\frac{x\pi-m\pi}{p}\right)\right)_{p\leqslant\sqrt{N}}\end{aligned} \tag{3.3.9}$$

Since $m\neq 0$, $\sin\left(\frac{x\pi}{p}\right)$ and $\sin\left(\frac{x\pi-m\pi}{p}\right)$ will not be zero at the same time. Then it shows that

$$\sin\left(\frac{x\pi}{p}\right)+\sin\left(\frac{x\pi-m\pi}{p}\right)\neq 0 \tag{3.3.10}$$

That is

$$\Theta\left(\sin\left(\frac{x\pi}{p}\right)+\sin\left(\frac{x\pi-m\pi}{p}\right)\right)_{p\leqslant\sqrt{N};p\nmid N;m\neq 0}=0 \tag{3.3.11}$$

By Theorem 14, it follows that

$$\begin{aligned}\Theta\left(\sin\left(\frac{x\pi}{p}\right)\sin\left(\frac{x\pi-m\pi}{p}\right)\right)_{p\leqslant\sqrt{N};p\nmid N;m\neq 0}&=\Theta\left(\sin\left(\frac{x\pi}{p}\right)\right)_{p\leqslant\sqrt{N}}+\Theta\left(\sin\left(\frac{x\pi-m\pi}{p}\right)\right)_{p\leqslant\sqrt{N}}\\&=2\Theta\left(\sin\left(\frac{x\pi}{p}\right)\right)_{p\leqslant\sqrt{N}}\end{aligned} \tag{3.3.12}$$

The theorem is thus proved.

Theorem 18. For Θ function, it follows that

$$\begin{aligned}&\Theta\left(\sin\left(\frac{x\pi}{p}\right)\sin\left(\frac{x\pi-m\pi}{p}\right)\sin\left(\frac{x\pi}{q}\right)\sin\left(\frac{x\pi-n\pi}{q}\right)\right)_{p\leqslant\sqrt{N};N\equiv m(\mathrm{mod}p);N\equiv n(\mathrm{mod}q)}\\&=\begin{cases}\Theta\left(\sin\left(\frac{x\pi}{p}\right)\sin\left(\frac{x\pi}{q}\right)\right)_{p\leqslant\sqrt{N};p\mid N;q\mid N}\\ 2\Theta\left(\sin\left(\frac{x\pi}{p}\right)\right)+2\Theta\left(\sin\left(\frac{x\pi}{q}\right)\right)-4\Theta\left(\sin\left(\frac{x\pi}{pq}\right)\right)_{p\leqslant\sqrt{N};p\nmid N;q\nmid N}\end{cases}\end{aligned} \tag{3.3.13}$$

Proof. When $p, q|N$, it shows that

$$\begin{aligned}&\underset{p\leqslant\sqrt{N};m=0;n=0)}{\Theta\left(\sin\left(\frac{x\pi}{p}\right)\sin\left(\frac{x\pi-m\pi}{p}\right)\sin\left(\frac{x\pi}{q}\right)\sin\left(\frac{x\pi-n\pi}{q}\right)\right)}\\ &=\underset{p\leqslant\sqrt{N};m=0;n=0)}{\Theta\left(\sin^2\left(\frac{x\pi}{p}\right)\sin^2\left(\frac{x\pi}{q}\right)\right)}\end{aligned}\tag{3.3.14}$$

By Theorem 15, this theorem is proved.

When $p, q \nmid N$, it shows that

$$\Theta\left(\sin\left(\frac{x\pi}{p}\right)+\sin\left(\frac{x\pi}{q}\right)\right);\Theta\left(\sin\left(\frac{x\pi-m\pi}{p}\right)+\sin\left(\frac{x\pi-n\pi}{q}\right)\right);$$

$$\Theta\left(\sin\left(\frac{x\pi}{p}\right)+\sin\left(\frac{x\pi-n\pi}{q}\right)\right);\Theta\left(\sin\left(\frac{x\pi-m\pi}{p}\right)+\sin\left(\frac{x\pi}{q}\right)\right)\tag{3.3.15}$$

By Theorem 15 and 16, it shows that

$$\begin{aligned}&\underset{p\leqslant\sqrt{N};N\equiv m(\mathrm{mod}p);N\equiv n(\mathrm{mod}q)}{\Theta\left(\sin\left(\frac{x\pi}{p}\right)\sin\left(\frac{x\pi-m\pi}{p}\right)\sin\left(\frac{x\pi}{q}\right)\sin\left(\frac{x\pi-n\pi}{q}\right)\right)}\\ &=\underset{p\leqslant\sqrt{N};p\nmid N;q\nmid N}{2\Theta\left(\sin\left(\frac{x\pi}{p}\right)\right)+2\Theta\left(\sin\left(\frac{x\pi}{q}\right)\right)-4\Theta\left(\sin\left(\frac{x\pi}{pq}\right)\right)}\end{aligned}\tag{3.3.16}$$

This theorem is thus proved.

Theorem 19. When $p_i|N$, use $\widehat{\varpi}(N)$ to denote the number of composite pairs of Ξ function. Then

$$\underset{p_i|N,p_i\leqslant\sqrt{N}}{\widehat{\varpi}(N)}-\pi(\sqrt{N})=\underset{p_i|N,p_i\leqslant\sqrt{N}}{\overrightarrow{\varpi}(N)}=\underset{p_i|N,p_i\leqslant\sqrt{N}}{\overleftarrow{\varpi}(N)}\tag{3.3.17}$$

That is

$$\underset{p_i|N,p_i\leqslant\sqrt{N}}{\widehat{\varpi}(N)}=\underset{p_i|N,p_i\leqslant\sqrt{N},l=\pi(\sqrt{N})}{\sum_{x=1}^{N}\Theta\left(\prod_{i=1}^{l}\sin\left(\frac{x\pi}{p_i}\right)\right)}\tag{3.3.18}$$

Wherein the composite pair refers to an order of one or two composites that correspond to the order of $\overrightarrow{\Gamma}$ and $\overleftarrow{\Gamma}$.

Proof. By Theorem 17 and 18, since $p_i \mid N$, each p_i of $\overrightarrow{\varpi}(N)$ and $\overleftarrow{\varpi}(N)$ will be repeated, then $p_i \leq \sqrt{N}$ will be repeated by $\underset{p_i|N}{\overrightarrow{\varpi}(N)}$; in the same way, $p_i \leq \sqrt{N}$ will be repeated by $\underset{p_i|N}{\overleftarrow{\varpi}(N)}$. Thus the theorem is proved.

Theorem 20. When $p_i \nmid N$, use $\tilde{\varpi}(N)$ to denote the number of composite pairs of $\Xi(N)$ function. Then

$$\tilde{\varpi}(N)=\underset{p_i\leqslant\sqrt{N},l=\pi(\sqrt{N})p_i\nmid N,N\equiv m_i(\mathrm{mod}p_i)}{\sum_{x=1}^{N}\Theta\left(\prod_{i=1}^{l}\sin\left(\frac{x\pi}{p_i}\right)\sin\left(\frac{x\pi-m_i\pi}{p_i}\right)\right)}\tag{3.3.19}$$

Proof. By Theorem 17 and 18

$$\begin{aligned}\widetilde{\varpi}(N) &= 2^1 \sum_{\substack{i=1 \\ p_i \leqslant \sqrt{N}; p_i \nmid N; l=\pi(\sqrt{N})}}^{l} \sum_{x=1}^{N} \Theta\left(\sin\left(\frac{x\pi}{p_i}\right)\right) - 2^2 \sum_{\substack{i=1 \\ p_i,p_j \leqslant \sqrt{N}; p_i,p_j \nmid N; p_ip_j \leqslant N; l=\pi(\sqrt{N})}}^{l-1} \sum_{j=i+1}^{l} \sum_{x=1}^{N} \Theta\left(\sin\left(\frac{x\pi}{p_ip_j}\right)\right) \\ &+ 2^3 \sum_{\substack{i=1 \\ p_i,p_j,p_k \leqslant \sqrt{N}; p_i,p_j,p_k \nmid N; p_ip_jp_k \leqslant N; l=\pi(\sqrt{N})}}^{l-2} \sum_{j=i+1}^{l-1} \sum_{k=j+1}^{l} \sum_{x=1}^{N} \Theta\left(\sin\left(\frac{x\pi}{p_ip_jp_k}\right)\right) - \cdots \\ &= \sum_{x=1}^{N} \Theta\left(\prod_{\substack{i=1 \\ p_i \leqslant \sqrt{N}; p_i \nmid N; N \equiv m_i (\bmod p_i); l=\pi(\sqrt{N})}}^{l} \sin\left(\frac{x\pi}{p_i}\right) \sin\left(\frac{x\pi - m_i\pi}{p_i}\right)\right)\end{aligned} \tag{3.3.20}$$

This theorem is thus proved.

Theorem 21. Use $\overset{\rightleftharpoons}{\varpi}(x)$ to denote the composites of the interval of $(\sqrt{N}, N-\sqrt{N})$, it follows that

$$\overset{\rightleftharpoons}{\varpi}(x) = \sum_{x=\sqrt{N}}^{N-\sqrt{N}} \Theta\left(\prod_{\substack{i=1 \\ p_i \leqslant \sqrt{N}; N \equiv m_i (\bmod p_i); l=\pi(\sqrt{N})}}^{l} \sin\left(\frac{x\pi}{p_i}\right) \sin\left(\frac{x\pi - m_i\pi}{p_i}\right)\right) \tag{3.3.21}$$

Proof. By the definition of composite pair of Ξ function, it shows that

$$\overset{\rightleftharpoons}{\varpi}(x) = \widehat{\varpi}(x) + \widetilde{\varpi}(x) \tag{3.3.22}$$

By Theorem 19 and 20, this theorem is proved.

Theorem 22. Use $\overset{\rightleftharpoons}{\pi}(x)$ to denote the number of prime pairs of the interval of $(\sqrt{N}, N-\sqrt{N})$, then

$$\overset{\rightleftharpoons}{\pi}(x) = (N - 2\sqrt{N}) - \sum_{x=\sqrt{N}}^{N-\sqrt{N}} \Theta\left(\prod_{\substack{i=1 \\ p_i \leqslant \sqrt{N}; N \equiv m_i (\bmod p_i); l=\pi(\sqrt{N})}}^{l} \sin\left(\frac{x\pi}{p_i}\right) \sin\left(\frac{x\pi - m_i\pi}{p_i}\right)\right) \tag{3.3.23}$$

Proof. Since $\overset{\rightleftharpoons}{\varpi}(x)$ of Theorem 21 includes the composite pairs of the interval $(\sqrt{N}, N-\sqrt{N})$, by the definition of Ξ function,we have

$$\overset{\rightleftharpoons}{\varpi}(x) = (N - 2\sqrt{N}) - \overset{\rightleftharpoons}{\varpi}(x) \tag{3.3.24}$$

Hence the theorem is proved.

4. Further analysis

Theorem 23. When N is an even composite, use $\pi(N)$ to denote the number of primes smaller than N, then

$$\pi(N) = \sum_{x=1}^{N} \Theta\left(\Theta\left(\prod_{\substack{i=1 \\ p_i \leqslant \sqrt{N}; l=\pi(\sqrt{N})}}^{l} \sin\left(\frac{x\pi}{p_i}\right)\right)\right) + \pi(\sqrt{N}) - 1 \tag{4.0.25}$$

Proof. By Theorem 10, it shows that $\sum_{x=1}^{N} \Theta\left(\prod_{\substack{i=1 \\ p_i \leqslant \sqrt{N}; l=\pi(\sqrt{N})}}^{l} \sin\left(\frac{x\pi}{p_i}\right)\right)$ includes the number of composites except $\pi(\sqrt{N})$. When $\prod_{\substack{i=1 \\ p_i \leqslant \sqrt{N}; l=\pi(\sqrt{N})}}^{l} \sin\left(\frac{x\pi}{p_i}\right) \neq 0$, by the definition of prime, x must be a prime. Then

$$\Theta\left(\prod_{\substack{i=1 \\ p_i \leqslant \sqrt{N}; x \leqslant N}}^{l} \sin\left(\frac{x\pi}{p_i}\right)\right) = 0 \tag{4.0.26}$$

Again, by Definition 1, it follows that

$$\Theta\left(\Theta\left(\prod_{\substack{i=1 \\ p_i \leqslant \sqrt{N}; l=\pi(\sqrt{N}); x \leqslant N}}^{l} \sin\left(\frac{x\pi}{p_i}\right)\right)\right) = 1 \tag{4.0.27}$$

Thus the theorem is proved.

Theorem 24. Let

$$\underset{x \leqslant N}{a_x} = \prod_{\substack{i=1 \\ p_i \leqslant \sqrt{N}; l=\pi(\sqrt{N})}}^{l} \sin\left(\frac{x\pi}{p_i}\right), \ \underset{x \leqslant N}{b_x} = \prod_{\substack{i=1 \\ p_i \leqslant \sqrt{N}; N \equiv m_i (\bmod p_i); l=\pi(\sqrt{N})}}^{l} \sin\left(\frac{x\pi - m_i\pi}{p_i}\right)$$

for Ξ function, when $a_x \neq 0$ and $b_x \neq 0$, it shows that

$$\Theta\left(\Theta\left(a_x b_x\right)\right) = \Theta\left(\Theta\left(a_x\right)\right)\Theta\left(\Theta\left(b_x\right)\right) \tag{4.0.28}$$

Proof. When $a_x \neq 0$ and $b_x \neq 0$, then $a_x b_x \neq 0$. It follows that

$$\Theta\left(a_x b_x\right) = 0 \Longrightarrow \Theta\left(\Theta\left(a_x b_x\right)\right) = 1 \tag{4.0.29}$$

$$a_x b_x \neq 0 \Rightarrow a_x \neq 0, b_x \neq 0 \Rightarrow \begin{array}{l} \Theta\left(a_x\right) = 0, \Theta\left(a_x\right) = 0 \\ \Theta\left(\Theta\left(a_x\right)\right) = 1; \Theta\left(\Theta\left(b_x\right)\right) = 1 \\ \Theta\left(\Theta\left(a_x\right)\right)\Theta\left(\Theta\left(b_x\right)\right) = 1 \end{array} \tag{4.0.30}$$

The theorem is thus proved.

Theorem 25. For an even composite N, the number of prime pairs (Goldbach Conjecture) of interval $(\sqrt{N}, N-\sqrt{N})$ is

$$\overrightarrow{\overleftarrow{\pi}}(x) = \sum_{x=\sqrt{N}}^{N-\sqrt{N}} \Theta\left(\Theta\left(\prod_{i=1}^{l} \sin\left(\frac{x\pi}{p_i}\right)\right)\right)\Theta\left(\Theta\left(\prod_{\substack{i=1 \\ p_i \leqslant \sqrt{N}; N \equiv m_i (\bmod p_i); l=\pi(\sqrt{N}}}^{l} \sin\left(\frac{x\pi - m_i\pi}{p_i}\right)\right)\right) \tag{4.0.31}$$

Proof. By the same way of theorem 23, theorem 22 shows that

$$\begin{aligned} \overrightarrow{\overleftarrow{\pi}}(x) &= \sum_{x=\sqrt{N}}^{N-\sqrt{N}} \Theta\left(\Theta\left(\prod_{\substack{i=1 \\ p_i \leqslant \sqrt{N}; N \equiv m_i (\bmod p_i); l=\pi(\sqrt{N})}}^{l} \sin\left(\frac{x\pi}{p_i}\right)\sin\left(\frac{x\pi - m_i\pi}{p_i}\right)\right)\right) \\ &= \sum_{x=\sqrt{N}}^{N-\sqrt{N}} \Theta\left(\Theta\left(\prod_{i=1}^{l} \sin\left(\frac{x\pi}{p_i}\right)\prod_{\substack{i=1 \\ p_i \leqslant \sqrt{N}; N \equiv m_i (\bmod p_i); l=\pi(\sqrt{N})}}^{l} \sin\left(\frac{x\pi - m_i\pi}{p_i}\right)\right)\right) \end{aligned} \tag{4.0.32}$$

Let

$$\prod_{\substack{i=1\\ p_i\leqslant\sqrt{N};l=\pi(\sqrt{N})}}^{l}\sin\left(\frac{x\pi}{p_i}\right)=a_x,\quad \prod_{\substack{i=1\\ p_i\leqslant\sqrt{N};N\equiv m_i(\mathrm{mod}p_i);l=\pi(\sqrt{N})}}^{l}\sin\left(\frac{x\pi-m_i\pi}{p_i}\right)\qquad =b_x \tag{4.0.33}$$

By Theorem 24, theorem 25 is proved.
Theorem 26. For a sufficiently large even composite of N, it shows that

$$\overset{\rightleftharpoons}{\pi}(N)>\frac{N-4\sqrt{N}}{\ln^2(N-\sqrt{N})} \tag{4.0.34}$$

Proof. Since

(4.0.35)
$$\begin{aligned}
\overset{\rightleftharpoons}{\pi}(x) &= \sum_{x=\sqrt{N}}^{N-\sqrt{N}}\Theta\left(\Theta\left(\prod_{\substack{i=1\\ p_i\leqslant\sqrt{N});l=\pi(\sqrt{N})}}^{l}\sin\left(\frac{x\pi}{p_i}\right)\right)\right)\Theta\left(\Theta\left(\prod_{\substack{i=1\\ p_i\leqslant\sqrt{N};N\equiv m_i(\mathrm{mod}p_i);l=\pi(\sqrt{N})}}^{l}\sin\left(\frac{x\pi-m_i\pi}{p_i}\right)\right)\right)\\
&= \sum_{x=\sqrt{N}}^{N-\sqrt{N}}\Theta\left(\Theta\left(\prod_{\substack{i=1\\ p_i\leqslant\sqrt{N});l=\pi(\sqrt{N})}}^{l}\sin\left(\frac{x\pi}{p_i}\right)\right)\right)\Theta\left(\Theta\left(\prod_{\substack{i=1\\ 2<p_i\leqslant\sqrt{N};N\equiv m_i(\mathrm{mod}p_i);l=\pi(\sqrt{N})}}^{l}\sin\left(\frac{x\pi-m_i\pi}{p_i}\right)\right)\right)\\
&= \sum_{x=\sqrt{N}}^{N-\sqrt{N}}\Theta\left(\Theta\left(\prod_{\substack{i=1\\ p_i\leqslant\sqrt{N});l=\pi(\sqrt{N})}}^{l}\sin\left(\frac{x\pi}{p_i}\right)\right)\right)\Theta\left(\Theta\left(\prod_{p_i\leqslant\sqrt{N};p_i\nmid N;N\equiv m_i(\mathrm{mod}p_i);l=\pi(\sqrt{N})}_i=1^l sin\left(\frac{x\pi-m_i\pi}{p_i}\right)\right)\right)
\end{aligned}$$

Let

$$a_x=\frac{\Theta\left(\Theta\left(\prod_{\substack{i=1\\ p_i\leqslant\sqrt{N};l=\pi(\sqrt{N})}}^{l}\sin\left(\frac{x\pi}{p_i}\right)\right)\right)}{N-2\sqrt{N}} \tag{4.0.36}$$

$$b_x=\frac{\Theta\left(\Theta\left(\prod_{\substack{i=1\\ p_i\leqslant\sqrt{N};N\equiv m_i(\mathrm{mod}p_i);l=\pi(\sqrt{N})}}^{l}\sin\left(\frac{x\pi-m_i\pi}{p_i}\right)\right)\right)}{N-2\sqrt{N}} \tag{4.0.37}$$

$$c_x=\frac{\Theta\left(\Theta\left(\prod_{\substack{i=1\\ p_i\leqslant\sqrt{N};l=\pi(\sqrt{N})}}^{l}\sin\left(\frac{x\pi}{p_i}\right)\right)\right)\Theta\left(\Theta\left(\prod_{\substack{i=1\\ p_i\leqslant\sqrt{N};N\equiv m_i(\mathrm{mod}p_i);l=\pi(\sqrt{N})}}^{l}\sin\left(\frac{x\pi-m_i\pi}{p_i}\right)\right)\right)}{N-2\sqrt{N}} \tag{4.0.38}$$

By Abel's Theorem (Multiplication of Series) [9], when $n \to \infty$ it follows that
(4.0.39)

$$\overrightarrow{\overleftarrow{\pi}}(x) = (N - 2\sqrt{N}) \sum_{x=\sqrt{N}}^{N-\sqrt{N}} \frac{\underset{p_i \leqslant \sqrt{N}; l=\pi(\sqrt{N})}{\Theta\left(\Theta\left(\prod_{i=1}^{l} \sin\left(\frac{x\pi}{p_i}\right)\right)\right)} \underset{p_i \leqslant \sqrt{N}; N \equiv m_i(\mathrm{mod} p_i); l=\pi(\sqrt{N})}{\Theta\left(\Theta\left(\prod_{i=1}^{l} \sin\left(\frac{x\pi - m_i\pi}{p_i}\right)\right)\right)}}{N - 2\sqrt{N}}$$

$$= (N - 2\sqrt{N}) \sum_{x=\sqrt{N}}^{N-\sqrt{N}} \frac{\underset{p_i \leqslant \sqrt{N}); l=\pi(\sqrt{N})}{\Theta\left(\Theta\left(\prod_{i=1}^{l} \sin\left(\frac{x\pi}{p_i}\right)\right)\right)}}{N - 2\sqrt{N}} \sum_{x=\sqrt{N}}^{N-\sqrt{N}} \frac{\underset{p_i \leqslant \sqrt{N}; N \equiv m_i(\mathrm{mod} p_i)}{\Theta\left(\Theta\left(\prod_{i=1}^{l} \sin\left(\frac{x\pi - m_i\pi}{p_i}\right)\right)\right)}}{N - 2\sqrt{N}}$$

$$= \frac{\sum_{x=\sqrt{N}}^{N-\sqrt{N}} \underset{p_i \leqslant \sqrt{N}); l=\pi(\sqrt{N})}{\Theta\left(\Theta\left(\prod_{i=1}^{l} \sin\left(\frac{x\pi}{p_i}\right)\right)\right)} \sum_{x=\sqrt{N}}^{N-\sqrt{N}} \underset{p_i \leqslant \sqrt{N}; N \equiv m_i(\mathrm{mod} p_i); l=\pi(\sqrt{N})}{\Theta\left(\Theta\left(\prod_{i=1}^{l} \sin\left(\frac{x\pi - m_i\pi}{p_i}\right)\right)\right)}}{(N - 2\sqrt{N})}$$

Since N is an even number, it surely has $2|N$.

Therefore
(4.0.40)

$$\sum_{x=\sqrt{N}}^{N-\sqrt{N}} \Theta\left(\Theta\left(\prod_{\substack{i=1 \\ p_i \leqslant \sqrt{N}; p_i \nmid N; N \equiv m_i(\mathrm{mod} p_i)}}^{l} \sin\left(\frac{x\pi - m_i\pi}{p_i}\right)\right)\right) \geqslant \sum_{x=\sqrt{N}}^{N-\sqrt{N}} \Theta\left(\Theta\left(\prod_{\substack{i=1 \\ 2 < p_i \leqslant \sqrt{N}; N \equiv m_i(\mathrm{mod} p_i)}}^{l} \sin\left(\frac{x\pi - m_i\pi}{p_i}\right)\right)\right)$$

$$> \sum_{x=\sqrt{N}}^{N-\sqrt{N}} \Theta\left(\Theta\left(\prod_{\substack{i=1 \\ p_i \leqslant \sqrt{N}; N \equiv m_i(\mathrm{mod} p_i)}}^{l} \sin\left(\frac{x\pi - m_i\pi}{p_i}\right)\right)\right)$$

So we have
(4.0.41)

$$\overrightarrow{\overleftarrow{\pi}}(x) > \frac{\sum_{x=\sqrt{N}}^{N-\sqrt{N}} \underset{p_i \leqslant \sqrt{N}); l=\pi(\sqrt{N})}{\Theta\left(\Theta\left(\prod_{i=1}^{l} \sin\left(\frac{x\pi}{p_i}\right)\right)\right)} \sum_{x=\sqrt{N}}^{N-\sqrt{N}} \underset{p_i \leqslant \sqrt{N}; N \equiv m_i(\mathrm{mod} p_i); l=\pi(\sqrt{N})}{\Theta\left(\Theta\left(\prod_{i=1}^{l} \sin\left(\frac{x\pi - m_i\pi}{p_i}\right)\right)\right)}}{(N - 2\sqrt{N})}$$

And
(4.0.42)

$$\sum_{x=\sqrt{N}}^{N-\sqrt{N}} \Theta\left(\Theta\left(\prod_{\substack{i=1 \\ p_i \leqslant \sqrt{N}}}^{l} \sin\left(\frac{x\pi}{p_i}\right)\right)\right) = \sum_{x=\sqrt{N}}^{N-\sqrt{N}} \Theta\left(\Theta\left(\prod_{\substack{i=1 \\ p_i \leqslant \sqrt{N}; N \equiv m_i(\mathrm{mod} p_i)}}^{l} \sin\left(\frac{x\pi - m_i\pi}{p_i}\right)\right)\right)$$

So

$$\overrightarrow{\overleftarrow{\pi}}(x) > \frac{1}{(N - 2\sqrt{N})} \left(\sum_{x=\sqrt{N}}^{N-\sqrt{N}} \Theta\left(\Theta\left(\prod_{\substack{i=1 \\ p \leqslant \sqrt{N}); l=\pi(\sqrt{N})}}^{l} \sin\left(\frac{x\pi}{p_i}\right)\right)\right) \right)^2 \tag{4.0.43}$$

When N is sufficiently large, by theorem 2, theorem 23 can be revised as

$$\pi(x) = \sum_{x=2}^{N} \Theta\left(\Theta\left(\prod_{\substack{i=1 \\ p_i \leqslant \sqrt{N}); l=\pi(\sqrt{N})}}^{l} \sin\left(\frac{x\pi}{p_i}\right)\right)\right) + \pi(\sqrt{N}) = \frac{N}{\ln N} \tag{4.0.44}$$

Therefore

$$\sum_{x=2}^{N}\Theta\left(\Theta\left(\prod_{\substack{i=1\\ p_i\leqslant\sqrt{N});l=\pi(\sqrt{N})}}^{l}\sin\left(\frac{x\pi}{p_i}\right)\right)\right)=\frac{N}{\ln N}-\frac{\sqrt{N}}{\ln\sqrt{N}}=\frac{N-2\sqrt{N}}{\ln N} \tag{4.0.45}$$

Therefore, the interval of $(\sqrt{N}, N-\sqrt{N})$ shows that

$$\pi'(N)=\sum_{x=\sqrt{N}}^{N-\sqrt{N}}\Theta\left(\Theta\left(\prod_{\substack{i=1\\ p_i\leqslant\sqrt{N});l=\pi(\sqrt{N})}}^{l}\sin\left(\frac{x\pi}{p_i}\right)\right)\right)+\pi(\sqrt{N})=\frac{N-3\sqrt{N}}{\ln(N-\sqrt{N})} \tag{4.0.46}$$

Therefore, when N is sufficiently large, it follows that

$$\overset{\rightrightarrows}{\pi}(x)>\frac{N-4\sqrt{N}+\frac{N}{N-2\sqrt{N}}}{\ln^2(N-\sqrt{N})}>\frac{N-4\sqrt{N}}{\ln^2(N-\sqrt{N})} \tag{4.0.47}$$

5. Discussion

Equation (2.1.1) is a mathematical explanation of sieve method to Theorem 3. According to Equation (2.1.5), Equation (2.1.4) could mean the same as Equation (2.1.1). A model can be applied to explain Equation (2.1.4). See Figure (2).

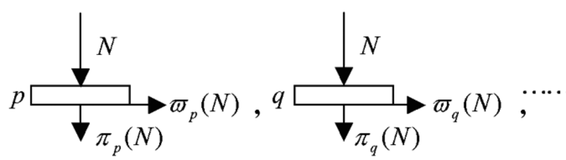

Figure 2.

In Figure (2), numbers such as $1, 2, \cdots, N$ will be sifted through the sieve $p, q, \cdots$ respectively. Here sift is served as a counting device, not a physical model. (When numbers have been sifted through the physical model, some numbers will be kept while others left out.) When numbers are being sifted through sieve p into q, the numbers remain the same. In order to derive accurate results from sieves p, q, etc, the issue of repetition between $\varpi_p(N)$ and $\varpi_q(N)$ should be taken into account.

Θ function can thus be introduced and together with the changing features of sine function, Equation (2.1.5) could be rewritten as Equation (2.3.28). Comparing Equation (2.3.28) with Equation (2.1.5), we know that Equation (2.3.28) has transformed Equation (2.1.5) into a physical sieve. See Figure (3).

When physical sieve is employed, numbers that could be divided exactly by p will be left out. The original numbers will be different from the numbers left after being sifted by p. Thus the numbers sifted by p will not be repeated by the numbers sifted by q. Moreover, the number obtained from the remaining ones should be $\pi(N)$. According to Equation (2.1.5) and (2.3.28), we may be able to get a prime number distribution Equation (3.1.1) which is represented by Θ function of sine function.

Goldbach Conjecture is a diophantine equation in form. For all positive whole numbers, when Goldbach Conjecture is presented under the condition that N is an even number larger than 6, the key to equation $N = x + y$ would be at least

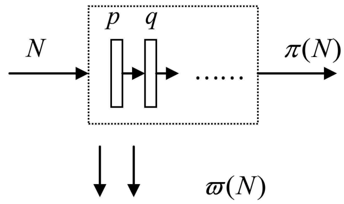

FIGURE 3.

one pair of prime numbers at the same time, such as x and y. The introduction of Ξ function is to account for a solution to an indefinite equation. We may use an ordered function to explain the following issue

$$(1, N-1)_1, (1, N-2)_2, \cdots, (k, N-k)_k, \cdots, (N-1, 1)_N$$

Equations (3.1.7) and (3.1.8) tell that "when numbers go through the physical sieve of Figure (3), the result will be related to the composition of numbers, but irrelated to the order of numbers, i.e. when numbers of $(1, 2, \cdots, N)$ and $(N, N-1, \cdots, 1)$ go through the same physical sieve, the outcome would remain the same. This can be expressed as $\overrightarrow{\Gamma}(z) = x$, $\overleftarrow{\Gamma}(z) = N - x = y$.

Equation (3.2.1) is one of the key concepts in this paper. By Figure (1), Equation (3.2.1) can illustrate such an issue that the sieve to the ordered solution set of $\overrightarrow{\Gamma}(z)$, and $\overleftarrow{\Gamma}(z)$ of Goldbach Conjecture will turn out to be the issue of simply sifting the prime numbers to the ordered solution set of $\overrightarrow{\Gamma}(z)$, and $\overleftarrow{\Gamma}(z)$.

Equation (3.3.18) tells us that when $p, q, \cdots$ can divide N exactly, and if x of the indefinite equation $N = x + y$ can be divided by $p, q, \cdots$ exactly, then the corresponding $y = N - x$ can also be divided by $p, q, \cdots$ exactly.

Equation (3.3.21) has actually constructed such a physical sieve that

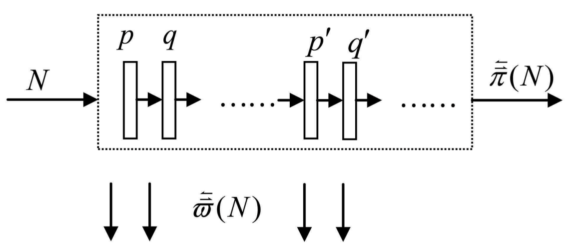

FIGURE 4.

The relationship between the sieve of $p, q, \cdots$ and that of p$p', q', \cdots$ would be that when N can be divided exactly by p, then p' and p are equivalent in figure (4). It's just a repetitive sieve which will not affect the result of sifting. Equation (3.2.5) serves as an example of this feature. When p cannot be divided exactly by N, p' and p are two different siftings. Equations (3.3.21) and (2.3.28) have no substantial difference. In the sieve built by Equation (3.3.21), the possible prime pairs could be

the ones that have passed the sifting of $p, q, \cdots$ (in the ordered solution set of $\overrightarrow{\Gamma}(x)$, x is a prime number); the composite number possible prime pairs that cannot pass the sieve will be sifted again by $p', q', \cdots$ (according to Definition 2, if the solution to an indefinite equation is a composite number, then it is the solution to the composite number pair). The pairs that cannot pass the sieve of $p', q', \cdots$ will be "false" prime pairs (in the ordered collection of $\overrightarrow{\Gamma}(x)$, x is a prime number while $y = N - x$ is a composite number. The ones that can pass the sieve of $p', q', \cdots$ will be real prime pairs (in the ordered solution set of $\overrightarrow{\Gamma}(x)$, x is a prime number, and $y = N - x$ is also a prime number).

Goldbach Conjecture can be interpreted as "truth without evidence", that is to say that under the premise that the conjecture is correct - the outcome is thus correct. All we have to do is to provide evidence to prove it.

Equation (4.0.28) is introduced on the basis of the features of Θ function ; with Equation (4.0.28), Equation (3.3.23) is turned into Equation (4.0.31). With Equation (4.0.31), the possibility to prove Equation (4.0.34) has become a reality. In Equation (4.0.31), the multiplication of progression has come out as a problem. Abel's Theorem is employed to solve this problem. It's well proved in Equation (4.0.34).

To sum up, Equation (4.0.34) can be testified for the following reasons:

a) It seems to be quite tough to explain the Goldbach Conjecture only by the conventional sieve since it involved the repetition issue when counting the numbers of sieves (prime factor). But a physical sieve has been set up (Equation (2.3.28)) by applying the sine function of Θ Function .

b) By introducing Ξ Function, the Goldbach Conjecture (the indefinite equation issue) can be simplified as a prime number sifting issue. Equations (3.3.21) and (2.3.28) have no substantial difference. By applying Equation (2.3.28), the composite and prime numbers have been sifted while through Equation (3.3.21), the pair numbers of composite and prime numbers are sifted.

c) With further application of Θ Function, Equation (4.0.31) is thus obtained, which is in line with the multiplication of progression issue. By applying Abel's Theorem, the correctness of Goldbach Conjecture is thus proved.

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Appendix

Example 1.(Theorem 4)

$$\pi(10) = 10 - 1 - \left\lceil \frac{10-2}{2} \right\rceil - \left\lceil \frac{10-3}{3} \right\rceil + \left\lceil \frac{10}{2\times 3} \right\rceil = 9 - 4 - 2 + 1 = 4$$

The fact is that $\pi(10)$ includes 2,3,5,7.

Example 2.(Theorem 5)

$$\varpi(10) = \left\lceil \frac{10-2}{2} \right\rceil + \left\lceil \frac{10-3}{3} \right\rceil - \left\lceil \frac{10}{2\times 3} \right\rceil = 4 + 2 - 1 = 5$$

The fact is that $\varpi(10)$ includes 4,6,8,9,10.

Example3. (Theorem 6) For integer 10, by eq.(6), it follows that

$$\varpi_3(10) = \left\lceil \frac{10-3}{3} \right\rceil = \left\lceil \frac{10}{3} \right\rceil - 1 = 2$$

$$\varpi_3(10) = \Theta\left(\sin\left(\frac{3\pi}{3}\right)\right) + \Theta\left(\sin\left(\frac{6\pi}{3}\right)\right) + \Theta\left(\sin\left(\frac{9\pi}{3}\right)\right) - 1 = 3 - 1 = 2$$

Example 4.(Theorem 8)

For integer 10, by formula (3), it follows that

$$\varpi_{2,3}(N) = \left\lceil \frac{10}{2} \right\rceil + \left\lceil \frac{10}{3} \right\rceil - \left\lceil \frac{10}{2\times 3} \right\rceil - 2 = 5$$

While by formula (4), it follows that

$$\begin{aligned}\varpi_{2,3}(N) = &\Theta\left(\sin\left(\frac{2\pi}{2}\right)\right) + \cdots + \Theta\left(\sin\left(\frac{10\pi}{2}\right)\right)\\ &+ \Theta\left(\sin\left(\frac{3\pi}{3}\right)\right) + \Theta\left(\sin\left(\frac{6\pi}{3}\right)\right) + \Theta\left(\sin\left(\frac{9\pi}{3}\right)\right)\\ &- \Theta\left(\sin\left(\frac{6\pi}{2\times 3}\right)\right) - 2 = 5 + 3 - 1 - 2 = 5\end{aligned}$$

And by formula (5), it shows that

$$\begin{aligned}\varpi_{2,3}(N) = &\Theta\left(\sin\left(\frac{2\pi}{2}\right)\sin\left(\frac{2\pi}{3}\right)\right) + \Theta\left(\sin\left(\frac{3\pi}{2}\right)\sin\left(\frac{3\pi}{3}\right)\right)\\ &+ \Theta\left(\sin\left(\frac{4\pi}{2}\right)\sin\left(\frac{4\pi}{3}\right)\right) + \Theta\left(\sin\left(\frac{6\pi}{2}\right)\sin\left(\frac{6\pi}{3}\right)\right)\\ &+ \Theta\left(\sin\left(\frac{8\pi}{2}\right)\sin\left(\frac{8\pi}{3}\right)\right) + \Theta\left(\sin\left(\frac{9\pi}{2}\right)\sin\left(\frac{9\pi}{3}\right)\right)\\ &+ \Theta\left(\sin\left(\frac{10\pi}{2}\right)\sin\left(\frac{10\pi}{3}\right)\right) - 2 = 7 - 2 = 5\end{aligned}$$

Example 5.(Theorem 12)

$$\begin{aligned}\overrightharpoon{\varpi}_3(10) &= \Theta\left(\sin\left(\frac{3\pi}{3}\right)\right) + \Theta\left(\sin\left(\frac{6\pi}{3}\right)\right) + \Theta\left(\sin\left(\frac{9\pi}{3}\right)\right) - 1\\ &= 3 - 1 = 2\end{aligned}$$

$$\begin{aligned}\overleftharpoon{\varpi}_3(10) &= \Theta\left(\sin\left(\frac{9\pi}{3}\right)\right) + \Theta\left(\sin\left(\frac{6\pi}{3}\right)\right) + \Theta\left(\sin\left(\frac{3\pi}{3}\right)\right) - 1\\ &= 3 - 1 = 2\end{aligned}$$

Example 6.(Theorem 17)

$$\begin{aligned}\sum_{m=1}^{2}\sum_{x=1}^{10}\Theta\left(\sin\left(\frac{x\pi - m\pi}{5}\right)\right) = &\Theta\left(\sin\left(\frac{4\pi-\pi}{5}\right)\right) + \Theta\left(\sin\left(\frac{9\pi-\pi}{5}\right)\right)\\ &+ \Theta\left(\sin\left(\frac{3\pi-2\pi}{5}\right)\right) + \Theta\left(\sin\left(\frac{8\pi-2\pi}{5}\right)\right) = 4\end{aligned}$$

$$2\sum_{x=1}^{10}\Theta\left(\sin\left(\frac{x\pi}{5}\right)\right)=2\left(\Theta\left(\sin\left(\frac{5\pi}{5}\right)\right)+\Theta\left(\sin\left(\frac{10\pi}{5}\right)\right)\right)=4$$

Example 7.(Theorem 19)

$$\begin{aligned}\widehat{\varpi}=&\Theta\left(\sin\left(\frac{2\pi}{2}\right)\sin\left(\frac{2\pi}{3}\right)\right)+\Theta\left(\sin\left(\frac{3\pi}{2}\right)\sin\left(\frac{3\pi}{3}\right)\right)+\Theta\left(\sin\left(\frac{4\pi}{2}\right)\sin\left(\frac{4\pi}{3}\right)\right)\\&+\Theta\left(\sin\left(\frac{6\pi}{2}\right)\sin\left(\frac{6\pi}{3}\right)\right)+\Theta\left(\sin\left(\frac{8\pi}{2}\right)\sin\left(\frac{8\pi}{3}\right)\right)\\&+\Theta\left(\sin\left(\frac{9\pi}{2}\right)\sin\left(\frac{9\pi}{3}\right)\right)+\Theta\left(\sin\left(\frac{12\pi}{2}\right)\sin\left(\frac{12\pi}{3}\right)\right)=7\end{aligned}$$

Example 8. (Theorem 25 and Theorem 26)

When $N=100$ and $\pi\sqrt{N}=4$, then $100\equiv 1(\mathrm{mod}3)$, $100\equiv 2(\mathrm{mod}7)$. The interval of $[10,90]$ shows that

When

$$\overrightarrow{\Gamma}(x)=\sum_{x=10}^{90}\Theta\left(\Theta\left(\sin\left(\frac{x\pi}{2}\right)\sin\left(\frac{x\pi}{3}\right)\sin\left(\frac{x\pi}{5}\right)\sin\left(\frac{x\pi}{7}\right)\right)\right)=1$$

x is $11,13,17,19,23,29,31,37,41,43,47,53,59,61,67,71,73,79,83,89$.

When

$$\overleftarrow{\Gamma}(x)=\sum_{x=10}^{90}\Theta\left(\Theta\left(\sin\left(\frac{x\pi}{2}\right)\sin\left(\frac{x\pi-\pi}{3}\right)\sin\left(\frac{x\pi}{5}\right)\sin\left(\frac{x\pi-2\pi}{7}\right)\right)\right)=1$$

x is 11,17,21,27,29,33,39,41,47,53,57,59,63,69,71,77,81,83,87,89.

$$\begin{aligned}&\Theta\left(\Theta\left(\sin^2\left(\tfrac{x\pi}{2}\right)\sin\left(\tfrac{x\pi}{3}\right)\sin^2\left(\tfrac{x\pi}{5}\right)\sin\left(\tfrac{x\pi}{7}\right)\sin\left(\tfrac{x\pi-\pi}{3}\right)\sin\left(\tfrac{x\pi-2\pi}{7}\right)\right)\right)\\=&\Theta\left(\Theta\left(\sin\left(\tfrac{x\pi}{2}\right)\sin\left(\tfrac{x\pi}{3}\right)\sin\left(\tfrac{x\pi}{5}\right)\sin\left(\tfrac{x\pi}{7}\right)\sin\left(\tfrac{x\pi-\pi}{3}\right)\sin\left(\tfrac{x\pi-2\pi}{7}\right)\right)\right)\end{aligned}$$

When

$$\overset{\rightleftharpoons}{\pi}(100)=\sum_{x=10}^{90}\Theta\left(\Theta\left(\sin\left(\frac{x\pi}{2}\right)\cdots\sin\left(\frac{x\pi-2\pi}{7}\right)\right)\right)=1$$

x is $11,17,29,41,47,53,59,71,83,89$.

$$\begin{aligned}\overset{\rightleftharpoons}{\pi}(100)=\sum_{x=10}^{90}&\Theta\left(\Theta\left(\sin\left(\frac{x\pi}{2}\right)\sin\left(\frac{x\pi}{3}\right)\sin\left(\frac{x\pi}{5}\right)\sin\left(\frac{x\pi}{7}\right)\right)\right)\\&\Theta\left(\Theta\left(\sin\left(\frac{x\pi}{2}\right)\sin\left(\frac{x\pi-\pi}{3}\right)\sin\left(\frac{x\pi}{5}\right)\sin\left(\frac{x\pi-2\pi}{7}\right)\right)\right)=1\end{aligned}$$

x is $11,17,29,41,47,53,59,71,83,89$.

$$\begin{aligned}\overset{\rightleftharpoons}{\pi}(100)=\sum_{x=10}^{90}&\Theta\left(\Theta\left(\sin\left(\frac{x\pi}{2}\right)\sin\left(\frac{x\pi}{3}\right)\sin\left(\frac{x\pi}{5}\right)\sin\left(\frac{x\pi}{7}\right)\right)\right)\\&\Theta\left(\Theta\left(\sin\left(\frac{x\pi-\pi}{3}\right)\sin\left(\frac{x\pi-2\pi}{7}\right)\right)\right)=1\end{aligned}$$

x is $11,17,29,41,47,53,59,71,83,89$.

The fact is that $\overset{\rightleftharpoons}{\pi}(100) = 10$ and the specific prime pairs are

$$11, 17, 29, 41, 47, 53, 59, 71, 83, 89$$
$$89, 83, 71, 59, 53, 47, 41, 29, 17, 11$$

When $N = 1000$ and $\pi\sqrt{N} = 11$, then $1000 \equiv 1(\text{mod}3)$, $1000 \equiv 6(\text{mod}7), \cdots$, $1000 \equiv 8(\text{mod}31)$. The interval of $[32, 968]$ has $\overset{\rightleftharpoons}{\pi}(1000) = 48$, the specific prime pairs are

$$47, 53, 59, \cdots, 443, 479, \cdots, 941, 947, 953$$
$$953, 947, 941, \cdots, 557, 521, \cdots, 59, 53, 47$$

When $N = 10000$ and $\pi\sqrt{N} = 25$, then $10000 \equiv 1(\text{mod}3)$,$10000 \equiv 4(\text{mod}7)$, $\cdots$,$10000 \equiv 9(\text{mod}97)$. The interval of $[100, 9900]$ has $\overset{\rightleftharpoons}{\pi}(10000) = 232$, the specific prime pairs are

$$113, 149, 163, \cdots, 4919, 5081, \cdots, 9837, 9851, 9887$$
$$9887, 9851, 9837, \cdots, 5081, 4919, \cdots, 163, 149, 113$$

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(Yifang Fan) Center for Scientific Research, Guangzhou Institute of Physical Education, Guangzhou 510500, P.R. China

E-mail address: `tfyf@gipe.edu.cn`

(Zhiyu Li) College of Foreign Languages, Jinan University, Guangzhou 510632, P.R. China